Stochastic analysis of Stravinsky’s varied ostinati

Daniel Brown
Department of Music, University of California at Santa Cruz, USA
dalabrow@ucsc.edu

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An analogy is drawn between the varied ostinato, a common musical device in Stravinsky's music, and Xenakis’ stochastically generated soundmasses. The analogy is constructed as follows: Stravinsky's varied ostinats are made up of cellular melodies; these cells can be represented as Markov chains, as can the sequence of cells themselves; the Markov chain for an sequence of cells has the property of ergodicity; this ergodicity allows two stochastic variables, the mean and variance of cell density in a sequence, to uniquely determine the chain. The irregularity of Stravinsky’s melodic repetition, in the past considered largely inscrutable, is perhaps accessible to musical analysis in terms of these stochastic variables. An example of computer-generated output that uses these variables to simulate Stravinsky’s style is given.

Irregularly varied repetition is one of the most salient features of Stravinsky's music, and also perhaps the most resistant to analysis. A varied ostinato is a device that Stravinsky used frequently; it features one or several small musical figures that are repeated irregularly—at varying time intervals, with varying durations, and in various orders. Boulez “cellular” analysis of Rite of Spring introduced a vivid terminology for describing the musical events being repeated irregularly; this notion of cells was further developed in subsequent analyses of Stravinsky by Messiaen and Jonathan Kramer. These analyses stop short, though, of analyzing the nature of the irregularity itself. In his analysis of Symphonies of Wind Instruments, for example, Kramer confesses that "there is little more I can do than show several sequences of durations where one might ordinarily find some regularity and then state that regularity is not there" (221).

Descriptions of cells—which use terms like of irregularity, randomness, permutations, etc.—depict them as stochastic objects, though the authors don't actually label them with this word. A cell, like a motive, is a small configuration of pitches and durations; the difference between a cell and a motive is that the order of pitches and durations in the cell is not fixed. However, there are usually certain orders which are privileged; as Kramer puts it, "after we have heard a number of sequences of the same cells, we understand which orderings are 'permissible' and which ones are not" (223). His term "sequence" refers to a section of music that contains a number of repetitions, also irregularly placed, of a cell or set of cells; a Stravinsky ostinato is thus a sequence. Kramer notes that "unpredictability within carefully defined boundary conditions exists on two adjacent hierarchic levels: between and within cells" (224). This suggests that, in order to distinguish the two types of “unpredictability,” there are two different random processes occurring in Stravinsky's ostinati, one governing the elements that make up the cells, and one governing the ordering of the cells themselves in a sequence. Furthermore, the notion of “privileged orders” suggests that the randomness in cells and sequences is constrained by some quantity or quantities.

This interpretation invites a method of analysis in which stochastic variables are derived that quantitatively measure some aspects of cells' higher-level collective behavior. The terminology and method for deriving such variables comes from Xenakis’ formulation of stochastically produced musical events. In Xenakis’ work, these events were soundmasses or “clouds.”

In this way, instances of cells in a cell sequence loosely resemble grains of sound in a Xenakian soundmass: each consists of small elements (cells or grains) that occur in a random manner; the overall sound quality of these elements is what gives the ostinato or cloud its character. In both cases, there are both individual and collective properties that shape the sound of the whole. In a cloud, the sonic character is a timbral result of the grains blending together to form a complex spectrum of partials. In a varied ostinato, it’s the highly syncopated rhythm and placement of cells that gives it a distinguishing quality. Even though clouds and sequences are topologically different, it’s not unintuitive to recognize some common qualities between the two; for instance,
the notion of cell density—the fraction of time that instances of the cell occur in a sequence—is similar to granular density in a cloud, the number of grains of some type that occur over a fraction of time.

![Diagram of grains and cells](image.png)

**Figure 1**

In this paper I will make this analogy into a more rigorous mapping by providing a method of quantitatively deriving the mean and variance of a cell’s density in a sequence. Furthermore, I’ll show that these two variables uniquely define a Markov chain for a cell sequence, and therefore are apt, distinguishing descriptors of the sequence. The reason I’ve chosen a Markov chain model is because of the difference between cells and grains, which is analogous to the difference between their natural counterparts. Grains are small: they generally last a fraction of a second, and blend with the other grains to the point of being indistinguishable. Cells are bigger; they endure perhaps several seconds, and contain discrete musical elements that maintain their identities over many repetitions. These elements are generally melodic, in the sense that some ordering principle is imposed upon the events.

Xenakis controlled both the individual sonic aspects of grains and their collective texture with stochastic variables. Regarding the collective aspect, he controlled the density of grains with algorithms like the ST algorithm, placing grains randomly on a grid in accordance with a given density and distribution. The internal melodic structure of cells, and the fact that they generally don’t overlap in Stravinsky’s ostinati, but are contiguous, preclude the use of similar methods to produce Stravinskian ostinati with desired cell densities. But a Markov model preserves cells’ ordering aspects, while still yielding stochastic information about the overall musical sound.

The collective aspect will be the aspect I consider, not the individual. But to do this, the smallest individual elements must first be modeled in order to get information from them that will then be used to determine the collective sonic qualities.

**Subcells, Cells, and Sequences**

Kramer defines three types of musical elements in Stravinsky’s ostinati: subcells, cells, and sequences. This order is hierarchic: subcells comprise cells, and cells comprise processes. As mentioned earlier, different types of randomness occur among these three layers. At times the distinction between these categories may seem blurry—that is, it can be hard to determine if a particular element is in one category or another. For my purposes, I differ slightly from Kramer’s (and Boulez’s, and Messiaen’s) categorization of elements. The following definitions of the three categories serve as the framework for the analyses to follow.
**Subcell:** A subcell is a set of one or more musical events whose order, if there are more than one event, is invariant. The duration of individual events can vary, however (in the following analyses, an “event” is simply a single note or chord).

**Cell:** A cell is a set of subcells. As a musical object, it is an abstract class; that is, an instance of a cell could feasibly consist of any contiguous set of instances of the cell’s subcells, including repetitions. In a given instance of a cell, there is no single fixed order in which its subcells occur, or a fixed number of subcells that need occur. However, the order and number of subcells in an instance of a cell also follow a probability distribution, which determines the “privileged” subcell orders Kramer remarked on. Cells are distinguished by their subcells. Stravinsky usually aids in the distinction through orchestration: subcells within a cell are similar in terms of register and instrumentation, and are different in both respects from subcells in other cells. I add one constraint to the definition of a cell: an instance of a cell always has a beginning subcell and an ending subcell; that is, every instance of a cell has a finite length.

**Sequence:** A sequence is a set of cells. Its relationship to its cells is analogous to a cell's relationship to its subcells; an instance of a sequence is a linear ordering of cell instances. Cell instances vary in their structure throughout a sequence, as described in the preceding paragraph. Stravinsky's varied ostinati are sequences. The structural difference between a sequence and a cell is that the sequence does not have a "stop" state. It is a process which, once begun, will continue indefinitely. The motivation for defining a sequence like this is intuitive: ostinati in Rite are long sections containing many irregular repetitions of a small number of cells, and thus don't convey a sense of beginning or ending. When a sequence ends, it sounds as if it was cut off, or interrupted by a new sequence.

Consider the string ostinato that occurs between rehearsal numbers 75 and 78 in "Dance of the Earth" in The Rite of Spring:

![Figure 2. "Dance of the Earth," The Rite of Spring, ostinato in strings](image)

To determine what constitute subcells in an ostinato, I choose the largest order-invariant units. Sometimes this is straightforward; other times the segmentation is subject to interpretation. Segmentation is an important notion which, unfortunately, goes beyond the scope of this paper. I have picked cells which are easy to segment, but not trivial; hopefully their usefulness in the model constructed justifies my interpretation.

The "Dance of the Earth" string ostinato can be segmented into five invariant figures of four sixteenth-notes each, each occurring on a beat, which constitute the entire ostinato. They are the subcells:

![Figure 3. Subcells in "Dance of the Earth" string ostinato](image)
Between rehearsal nos. 75 and 76, cell instances begin with subcell \( a_0 \) and end with subcell \( a_4 \). Between these beginning and ending subcells, the order of subcells is variable. The network diagram below depicts all the various orders of figures in the phrase. I include a "stop" node, which represents the end of a cell instance. The numbers on each edge represent the probability of that transition occurring, which are calculated directly from the score. By defining a cell as a network, a cell instance can be interpreted as a random walk (i.e., a Markov chain) over its subcells.

**Figure 4.** Network diagram for "Dance of the Earth" cell

**Markov Analysis**

For a cell with \( n \) subcells, define its transition matrix as an \((n+1) \times (n+1)\) matrix, where entry \( s_{ij} \) gives the transition probability from subcell \( i \) to subcell \( j \) for \( i,j \leq n \), and the entry \( s_{i(n+1)} \) gives the probability that subcell \( i \) will go to the "stop" state—i.e., that subcell \( i \) will be the last subcell in an instance. To calculate a transition matrix for a cell over a set of instances, sum the number of times each subcell \( i \) goes to each subcell \( j \) or the "stop" state, use that sum for the \( i,j \)th entry in the transition matrix, and then normalize each row of the matrix so that its entries sum to 1. The \((n+1)\)th row consists of all zeroes except for the last value: \( s_{(n+1)(n+1)} = 1 \). This designates the stop state as an absorbing state: once a Markov chain has reached the stop state, it stays there.

It can be shown that every cell modeled this way is an absorbing Markov chain; that is, any instance will reach the stop state eventually. Because of this, we can calculate the expected length of an instance of a cell from its matrix. Let \( t_i \) be the expected length of time to go from subcell \( i \) to the stop state. The vector \( t = [t_0 \ t_1 \ldots \ t_n] \) is calculated by solving the equation

\[
t = (I - Q)^{-1}b \tag{Eq. 1}
\]

where

- \( I \) is the \( n \times n \) identity matrix,
- \( Q \) is the matrix of transition probabilities between all subcells excluding the stop state
- \( b \) is the vector of expected subcell lengths \( b_j \) for each subcell \( j \).

The expected length of the cell is the dot product of \( t \) and the vector \( a \) of start-state probabilities, i.e. the probability that a chain will begin with subcell \( i \):
\[ E_{\text{length}(a)} = t \cdot a \]  
(Eq. 2)

From the network diagram given earlier, we have the following matrix:

\[
A = \begin{bmatrix}
  a_0 & a_1 & a_2 & a_3 & a_4 & \text{stop} \\
  a_0 & 0 & 1 & 0 & 0 & 0 \\
  a_1 & 0 & .06 & .78 & .06 & 0 .11 \\
  a_2 & 0 & 0 & 0 & 0 & .29 .71 \\
  a_3 & 0 & 1 & 0 & 0 & 0 \\
  a_4 & 0 & 0 & 0 & 0 & 1 \\
  \text{stop} & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Solving for equations 1 and 2 above gives

\[
t = [3.31 \ 2.31 \ 1.29 \ 3.31 \ 1.0]
\]

and

\[
t \cdot a = [3.31 \ 2.31 \ 1.29 \ 3.31 \ 1] \cdot [1 \ 0 \ 0 \ 0 \ 0] = 3.31
\]

The chain will generate cells with a mean length of 3.31 quarter-notes. This is in agreement with the mean length of the cell's instances in the score.

This models the distribution of the subcells in a cell; the distribution of the cells in a sequence uses another Markov matrix. In this particular ostinato, the period between cell instances always occurs in multiples of one quarter-note (filled in with a single repeating sixteenth-note by the horns). This, too, can be represented as a cell with a single subcell and an expected duration of 1; the "resting" state. Counting the transitions between the string cell and the resting state yields the following:

\[
P = \begin{bmatrix}
  \text{cell} & \text{rest} \\
  .4 & .6 \\
  .59 & .41 \\
\end{bmatrix}
\]

There is one important difference between the sequence matrix and the cell matrix. Unlike a cell, a cell sequence does not have a clear beginning or ending cell. It is a process which, once begun, will continue indefinitely. Stravinsky's ostinati are long sections containing many irregular repetitions of a small number of cells, and thus don't convey a sense of beginning or ending or movement. When a sequence ends, it usually sounds as if it was cut off arbitrarily, or was interrupted by a new sequence. The sequence's matrix thus has no stop state: it is non-absorbing.

The sequence matrix also has two more important properties. It is recurrent, meaning that when a chain leaves a cell, it will, with probability 1, return to that cell at some point. It is aperiodic, meaning the number of steps it takes to return to the cell it just left is not always the same, nor will it always be a multiple of some number greater than one (that is, there are no "hidden" cycles).

A Markov chain that is both recurrent and aperiodic is called an ergodic chain. An ergodic Markov chain is significant because of its steady-state probabilities. The steady-state probabilities of a Markov chain are the average amounts of time the chain will spend in each of its states--"over the long run." An ergodic chain will converge to its steady-state probability distribution regardless of the initial value of the chain. This means that the probability distribution, or any stochastic variables that define it, also completely define the chain itself--the ordering of events becomes
insignificant; only their distribution is important. This is different from the Markov chains that model cells, in which the order of subcells is important to some extent. The difference Kramer mentions between the “unpredictability… on two adjacent hierarchic levels” can now be represented in more precise terms: transition matrices for cells are non-ergodic, while those for cell sequences are ergodic. The ergodicity of a cell sequence matrix will allow the further derivation of stochastic variables that define the sequence.

To find the vector \( \mathbf{w} = [w_1, w_2, \ldots, w_n] \) consisting of the steady-state probability for each cell in a cell sequence matrix \( \mathbf{P} \), solve the equation

\[
\mathbf{w} \mathbf{P} = \mathbf{w} \quad \text{(Eq. 3)}
\]

The mean density of the \( i \)th cell is then given by

\[
\frac{w_i E_i}{\sum (w_j E_j)} \quad \text{(Eq. 4)}
\]

where \( E_j \) represents the expected duration of cell \( j \), and the sum in the denominator is taken over all the cells in the sequence.

From these equations, the mean densities of the string cell and the “rest” cell in the Dance of the Earth ostinato are determined to be .76 and .24, respectively. The variance of the densities can also be calculated; it will necessarily be the same for each cell, since cells in a sequence are contiguous. The transition matrix for a two-cell sequence can be represented in terms of two variables, as follows:

\[
\begin{bmatrix}
1 - p & p \\
q & 1 - q
\end{bmatrix} \quad \text{(Eq. 5)}
\]

The variance \( \sigma \) of the densities is then given by

\[
\sigma = \frac{2w_1(w_2 + q)}{(p - q)(w_2 - w_1)} + w_1 + w_1^2 \quad \text{(Eq. 6)}
\]

For the "Dance of the Earth" ostinato, the variance of the cell density is equal to .17.

Conversely, given a desired mean \( \mu \) and variance \( \sigma \) for the density of one cell, unique values for \( p \) and \( q \) can be derived:

\[
w_1 = \frac{\mu E_1}{\mu E_1 + (1 - \mu) E_2} \quad \text{(Eq. 7)}
\]

\[
w_2 = \frac{(1 - \mu) E_2}{\mu E_1 + (1 - \mu) E_2} \quad \text{(Eq. 8)}
\]

\[
p = \alpha w_1 \\
q = \alpha w_1 \quad \text{(Eq. 9, 10)}
\]

where \( \alpha = \frac{2w_1w_2}{\sigma + w_1w_2} \).

The transition matrix for a two-cell sequence is thus completely determined by these two stochastic variables, the mean and variance of the density of one of the cells. The two variables that govern sequences are analogues to the stochastic variables Xenakis used to control the overall sonic texture of his soundmasses. In other words, given two cells with defined transition matrices over their subcells, a unique transition matrix for the two-cell sequence can be calculated from any given mean and variance of one of the cells’ density. This uniqueness suggests that
these two variables can be used as descriptive quantities in musical analysis of music composed using cellular melodies.

**Application: computer-generated music in the style of Stravinsky**

It is reasonable to consider whether, after all this formal treatment, the quantities defined in this paper actually convey a significant musical quality. To test this, I have implemented the analytical techniques presented in this paper in a computer program (written in Python). The subcell pitch and rhythm data and transition matrices for two cells are loaded into the program. The cell density mean and variance of one of the cells can then be set arbitrarily; the program then calculates the matrix given in equation 5 and uses this matrix to output a sequence with the given values of these variables. I present a sample of the program’s output below.

The following excerpt (figure 5) from the “Sacrificial Dance” in *The Rite of Spring* (lasting from rehearsal number 192 to 196) consists of two cells, (labeled (“ascending” and a “descending”). The subcells and transition matrices associated with each cell are listed after the excerpt (figure 6). The mean cell density of the ascending cell is .24, and the variance of the density is .09. Figure 7 shows an example of the output of the program when the mean and variance of the ascending cell’s density are set to these two values. While different from the actual score, the quality of irregularity in the Stravinsky excerpt and the computer-generated output is very similar.

![Figure 5. Rite of Spring, "Sacrificial Dance" RN 192-195](image-url)
“Ascending” cell—subcells and transition matrix:

\[
\begin{bmatrix}
0 & 0.33 & 0.67 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

“Descending” cell—subcells and transition matrix:

\[
\begin{bmatrix}
0 & 0.12 & 0.25 & 0.5 & 0.13 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0.14 & 0 & 0 & 0 & 0.86 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Figure 6. “Sacrificial Dance” cells and subcell transition matrices, RN 192-195
Figure 7. Computer-generated output using cells from “Sacrificial Dance” with mean cell density = 0.24, cell density variance = 0.09
While the musical examples presented in this paper were all taken from (or simulate) Stravinsky’s works, the analytical techniques are solely based on the use of stochastic variables, as invented by Xenakis. It has been my hope to show that this notion of stochastic parameters can meaningfully apply to a larger class of musical elements than the timbral and textural aspects of traditional “stochastic” music, and subsequently to other composers’ styles in which cellular techniques are employed. Considering Stravinsky’s varied ostinati in terms of overall stochastic parameters opens up new avenues for both analysis and composition. In terms of analysis, it provides precise quantities that can be derived from the music, and which describe aspects of Stravinsky’s irregular repetition. The two quantities presented in this paper are cell density mean and variance; further research could derive more such variables. For composers, it offers a means of applying large-scale stochastic control to a musical parameter—melody—that is not typically associated with stochastic music.

References


